The volume of a closed hyperbolic 3-manifold is bounded by π times the length of any presentation of its fundamental group.

Daryl Cooper*

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Theorem 0.1 *Suppose M is a closed hyperbolic 3-manifold. Given a presentation of $\pi_1 M$ let L be the sum of the word-lengths of the relations and n the number of relations of length at least 3. Then volume$(M) < \pi(L-2n)$.*

Proof. A **pleated disc** is a map covered by a map of a disc into $\mathbb{H}^3$ such that there is a triangulation of the disc with vertices only on the boundary of the disc and with the property that the image of each 2-simplex is a geodesic 2-simplex in $\mathbb{H}^3$. A presentation of M gives a set of generators and relations. For simplicity, we will assume every relation has word length at least 3. This may be realized geometrically by a map $f : S \longrightarrow M$ of a 2-complex S which induces an isomorphism of $\pi_1 S$ onto $\pi_1 M$. The map f may be homotoped so that edges map to geodesics and f restricted to each 2-cell is a pleated disc. The area of a pleated disc is at most π times the number of 2-simplices. The boundary of a 2-cell, D, in S represents a relation, and the number of 2-simplices in D is the number of edges in ∂D minus 2. The number of edges in ∂D is the word length of the relation represented by ∂D. Thus the total surface area of $f(S)$ is at most $\pi(L-2n)$.

Let X be the closure of a component of $M - f(S)$, then X lifts to $\mathbb{H}^3$. For otherwise, there is a loop γ in X which is not contractible in M. Since S is mapped into $M - \gamma$ the isomorphism $f_* : \pi_1 S \longrightarrow \pi_1 M$ factors through $\pi_1(M-\gamma)$. Thus the composite

$$\pi_1 M \cong \pi_1 S \longrightarrow \pi_1(M-\gamma) \longrightarrow \pi_1 M$$

is the identity, where the second map is induced by inclusion. Now M is aspherical hence $\pi_2(M-\gamma) = 0$ because otherwise, by the sphere theorem, γ would be contained inside a ball thus contractible in M. Hence $M-\gamma$ is a $K(\pi,1)$ thus the first homomorphism is induced by a continuous map $M \longrightarrow (M-\gamma)$. Thus the composite

$$M \longrightarrow (M-\gamma) \longrightarrow M$$

is a π_1-isomorphism hence a homotopy equivalence. Consideration of the induced map on H_3 gives a contradiction:

$$H_3(M) \longrightarrow H_3(M-\gamma) \longrightarrow H_3(M)$$

since the composite is an isomorphism and M is closed.

The isoperimetric inequality (for example [1] p283) for $\mathbb{H}^3$ states that, for a given volume, the smallest ratio of surface area divided by volume is attained by a sphere. Computation shows that this ratio is always greater than 2. This asymptotic ratio is attained by a horosphere. Thus the surface area of the polyhedron X in $\mathbb{H}^3$ is at least 2 times its volume. Now S may be subdivided so that each 2-cell appears in exactly two such polyhedra, thus the total surface area of S is greater than 1 times the volume of M. Putting this together with the first part gives the result. □

In his thesis, Matt White [2] has obtained (a much deeper result) an explicit bound on the **diameter** of M in terms of the sum of the lengths of the relations. He also extends these results to the finite volume case.

References

[1] I. Chavel. *Riemannian Geometry: A Modern Introduction.* pub. Cambridge University Press (1993).

[2] M. White. UCSB Thesis, to appear.

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